ADDITIVITY OF JORDAN ELEMENTARY MAPS ON RINGS

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Abstract. We prove that Jordan elementary surjective maps on rings are automatically additive.

Elementary operators were originally introduced by Brešar and Šerml ([1]). In the last decade, elementary maps on operator algebras as well as on rings attracted more and more attentions. It is very interesting that elementary maps and Jordan elementary maps on some algebras and rings are automatically additive. The aim of this note is to continue to study the additivity of Jordan elementary maps on rings and standard operator algebras. We first define Jordan elementary maps as follows.

Definition 1. Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings, and let $M\colon \mathcal{R} \to \mathcal{R}'$ and $M^*\colon \mathcal{R}' \to \mathcal{R}$ be two maps. Call the ordered pair (M, M^*) a *Jordan elementary map* of $\mathcal{R} \times \mathcal{R}'$ if

$$\begin{cases} M(aM^*(x) + M^*(x)a) = M(a)x + xM(a), \\ M^*(M(a)x + xM(a)) = aM^*(x) + M^*(x)a \end{cases}$$

for all $a \in \mathcal{R}$, $x \in \mathcal{R}'$.

Note that the Jordan elementary maps defined above are different from those in [3].

We now introduce some definitions and results. Let $\mathcal{R}$ be a ring, if $a\mathcal{R}b = \{0\}$ implies either $a = 0$ or $b = 0$, then $\mathcal{R}$ is called a *prime* ring. A ring $\mathcal{R}$ is said to be 2-*torsion free* if $2a = 0$ implies $a = 0$.

Suppose that $\mathcal{R}$ is a ring containing a nontrivial idempotent e_1. Let $e_2 = 1 - e_1$ (Note that $\mathcal{R}$ need not have an identity element). We set $\mathcal{R}_{ij} = e_i\mathcal{R}e_j$, for $i, j = 1, 2$. Then we may write $\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{12} \oplus \mathcal{R}_{21} \oplus \mathcal{R}_{22}$. It should be mentioned here that this significant idea is due to Martinadale ([6]) which has become a key tool in dealing with the additivity of a large number of maps on some rings and operator algebras. In what follows, a_{ij} will denote that $a_{ij} \in \mathcal{R}_{ij}$ $(1 \le i, j \le 2)$.

We denote by $B(X)$ the algebra of all linear bounded operators on a Banach space X. A subalgebra of $B(X)$ is called a *standard operator algebra* if it contains all finite rank operators in $B(X)$.

Now we are ready to state our main result of this note.

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Theorem 2. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings. Suppose that $\mathcal{R}$ is a 2-torsion free ring containing a nontrivial idempotent e_1 and satisfies*

(i) $e_iae_j\mathcal{R}e_k = \{0\}$, or $e_k\mathcal{R}e_iae_j = \{0\}$ implies $e_iae_j = 0$ $(1 \le i, j, k \le 2)$, where $e_2 = 1 - e_1$;

(ii) If $e_2ae_2be_2 + e_2be_2ae_2 = 0$ for each $b \in \mathcal{R}$, then $e_2ae_2 = 0$.

Suppose that $M\colon \mathcal{R} \to \mathcal{R}'$ and $M^*\colon \mathcal{R}' \to \mathcal{R}$ are surjective maps such that

$$\begin{cases} M(aM^*(x) + M^*(x)a) = M(a)x + xM(a), \\ M^*(M(a)x + xM(a)) = aM^*(x) + M^*(x)a \end{cases}$$

for all $a \in \mathcal{R}$, $x \in \mathcal{R}'$. Then both M and M^* are additive.

The proof of this theorem is organized as a series of lemmas. We begin with

Lemma 3. *$M(0) = 0$ and $M^*(0) = 0$.*

Proof. We have $M(0) = M(0M^*(0) + M^*(0)0) = M(0)0 + 0M^*(0) = 0$.

Similarly, $M^*(0) = M^*(M(0)0 + 0M(0)) = 0M^*(0) + M^*(0)0 = 0$. □

The following lemma is very useful though the proof is simple.

Lemma 4. *Let $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathcal{R}$.*

(i) If $a_{ij}t_{jk} = 0$ for each $t_{jk} \in \mathcal{R}_{jk}$ $(1 \le i, j, k \le 2)$, then $a_{ij} = 0$.

Dually, if $t_{ki}a_{ij} = 0$ for each $t_{ki} \in \mathcal{R}_{ki}$ $(1 \le i, j, k \le 2)$, then $a_{ij} = 0$.

(ii) If $t_{ij}a + at_{ij} \in \mathcal{R}_{ij}$ for every $t_{ij} \in \mathcal{R}_{ij}$ $(1 \le i \ne j \le 2)$, then $a_{ji} = 0$

(iii) If $a_{ii}t_{ii} + t_{ii}a_{ii} = 0$ for every $t_{ii} \in \mathcal{R}_{ii}$ $(i = 1, 2)$, then $a_{ii} = 0$;

(iv) If $t_{jj}a + at_{jj} \in \mathcal{R}_{ij}$ for every $t_{jj} \in \mathcal{R}_{jj}$ $(1 \le i \ne j \le j)$, then $a_{ji} = 0$ and $a_{jj} = 0$.

Dually, if $t_{jj}a + at_{jj} \in \mathcal{R}_{ji}$ for every $t_{jj} \in \mathcal{R}_{jj}$ $(1 \le i \ne j \le j)$, then $a_{ij} = 0$ and $a_{jj} = 0$.

Proof. (i) It follows from condition (i) of Theorem 2 directly.

(ii) Since $t_{ij}a + at_{ij} \in \mathcal{R}_{ij}$, we have $(t_{ij}a + at_{ij})e_i = 0$. Thus, $t_{ij}ae_i = 0$, i.e., $t_{ij}a_{ji} = 0$. By (i), we have $a_{ji} = 0$.

(iii) For the case of $i = 1$, we have $0 = a_{11}e_1 + e_1a_{11} = a_{11} + a_{11} = 2a_{11}$, and so $a_{11} = 0$ since $\mathcal{R}$ is 2-torsion free.

The case of $i = 2$ is the same as condition (ii) of Theorem 2 as $\mathcal{R}$ is 2-torsion free.

(iv) From $t_{jj}a + at_{jj} \in \mathcal{R}_{ij}$, we have $(t_{jj}a + at_{jj})e_i = 0$. Then $t_{jj}a_{ji} = 0$, and so $a_{ji} = 0$.

Again, from $t_{jj}a + at_{jj} \in \mathcal{R}_{ij}$, we have $e_j(t_{jj}a + at_{jj})e_j = 0$, i.e., $t_{jj}a_{jj} + a_{jj}t_{jj} = 0$. By (iii), we have $a_{jj} = 0$. □

Lemma 5. *M and M^* are injective.*

Proof. First, we show that M is injective. Let $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$ be two elements of $\mathcal{R}$. Suppose that $M(a) = M(b)$.

For every $t_{ij} \in \mathcal{R}_{ij}$, by surjectivity of M^* there exists $x(i,j) \in \mathcal{R}'$ such that $M^*(x(i,j)) = t_{ij}$. We compute

$$\begin{aligned} t_{ij}a + at_{ij} &= M^*(x(i,j))a + aM^*(x(i,j)) = M^*(x(i,j)M(a) + M(a)x(i,j)) \\ &= M^*(x(i,j)M(b) + M(b)x(i,j)) = M^*(x(i,j))b + bM^*(x(i,j)) \\ &= t_{ij}b + bt_{ij}. \end{aligned}$$

Therefore, we have

$$t_{ij}a + at_{ij} = t_{ij}b + bt_{ij} \tag{1}$$

Letting $i = j = 2$ in the above equality, we have

$$t_{22}a_{21} + t_{22}a_{22} + a_{12}t_{22} + a_{22}t_{22} = t_{22}b_{21} + t_{22}b_{22} + b_{12}t_{22} + b_{22}t_{22}.$$

This implies that $t_{22}a_{21} = t_{22}b_{21}$, $a_{12}t_{22} = b_{12}t_{22}$, and $t_{22}a_{22} + a_{22}t_{22} = t_{22}b_{22} + b_{22}t_{22}$. By Lemma 4, we get $a_{21} = b_{21}$, $a_{12} = b_{12}$ and $a_{22} = b_{22}$.

If $i = 1$ and $j = 2$, then equality (1) becomes

$$t_{12}a_{21} + t_{12}a_{22} + a_{11}t_{12} + a_{21}t_{12} = t_{12}b_{21} + t_{12}b_{22} + b_{11}t_{12} + b_{21}t_{12},$$

and so $a_{11}t_{12} = b_{11}t_{12}$.Thus $a_{11} = b_{11}$. Therefore we can infer that M is injective.

We now show that M^* is injective. Let $x, y \in \mathcal{R}'$ such that $M^*(x) = M^*(y)$. Since M is a bijection, we may pick $a, b \in \mathcal{R}$ such that $a = M^{-1}(x)$ and $b = M^{-1}(y)$. We write $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$.

For each $t_{ij} \in \mathcal{R}_{ij}$, by the surjectivity of M^*M, there is a $c(i,j) \in \mathcal{R}$ such that $M^*M(c(i,j)) = t_{ij}$.

We consider

$$\begin{aligned} & t_{ij}a + at_{ij} \\ = \;& t_{ij}M^{-1}(x) + M^{-1}(x)t_{ij} \\ = \;& M^*M(c(i,j))M^{-1}(x) + M^{-1}(x)M^*M(c(i,j)) \\ = \;& M^*(M(c(i,j))MM^{-1}(x) + MM^{-1}(x)M(c(i,j))) \\ = \;& M^*(M(c(i,j))x + xM(c(i,j))) = c(i,j)M^*(x) + M^*(x)c(i,j) \\ = \;& c(i,j)M^*(y) + M^*(y)c(i,j) = M^*(M(c(i,j))y + yM(c(i,j))) \\ = \;& M^*(M(c(i,j))MM^{-1}(y) + MM^{-1}(y)M(c(i,j))) \\ = \;& M^*M(c(i,j))M^{-1}(y) + M^{-1}(y)M^*M(c(i,j)) \\ = \;& t_{ij}M^{-1}(y) + M^{-1}(y)t_{ij} \\ = \;& t_{ij}b + bt_{ij}, \end{aligned}$$

i.e., $t_{ij}a + at_{ij} = t_{ij}b + bt_{ij}$.

With the same argument above we can get $a_{11} = b_{11}, a_{12} = b_{12}$, $a_{21} = b_{21}$, and $a_{22} = b_{22}$. Hence $a = b$, equivalently, $x = y$, which completes the proof. $\square$

From the above lemma we see that both M and $M^{*^{-1}}$ are bijective.

Lemma 6. *The pair $(M^{*^{-1}}, M^{-1})$ is a Jordan elementary map on $\mathcal{R}\times\mathcal{R}'$. That is,*

$$\begin{cases} M^{*^{-1}}(aM^{-1}(x)+M^{-1}(x)a)=M^{*^{-1}}(a)x+xM^{*^{-1}}(a), \\ M^{-1}(M^{*^{-1}}(a)x+xM^{*^{-1}}(a))=aM^{-1}(x)+M^{-1}(x)a \end{cases}$$

for all $a\in\mathcal{R}$, $x\in\mathcal{R}'$.

Proof. We consider

$$\begin{aligned} M^*(M^{*^{-1}}(a)x+xM^{*^{-1}}(a)) &= M^*(M^{*^{-1}}(a)MM^{-1}(x)+MM^{-1}(x)M^{*^{-1}}(a)) \\ &= aM^{-1}(x)+M^{-1}(x)a, \end{aligned}$$

which leads to the first equality. The second one goes similarly. □

The following result will be used frequently in this note.

Lemma 7. *Let $a,b,c\in\mathcal{R}$ such that $M(c)=M(a)+M(b)$. Then*

$$M^{*^{-1}}(tc+ct)=M^{*^{-1}}(ta+at)+M^{*^{-1}}(tb+bt)$$

for all $t\in\mathcal{R}$

Proof. For every $t\in\mathcal{R}$, applying Lemma 6, we have

$$\begin{aligned} M^{*^{-1}}(tc+ct) &= M^{*^{-1}}(tM^{-1}M(c)+M^{-1}M(c)t)=M^{*^{-1}}(t)M(c)+M(c)M^{*^{-1}}(t) \\ &= M^{*^{-1}}(t)(M(a)+M(b))+(M(a)+M(b))M^{*^{-1}}(t) \\ &= (M^{*^{-1}}(t)M(a)+M(a)M^{*^{-1}}(t))+(M^{*^{-1}}(t)M(b)+M(b)M^{*^{-1}}(t)) \\ &= M^{*^{-1}}(ta+at)+M^{*^{-1}}(tb+bt). \end{aligned}$$

□

Lemma 8. *Let $a_{ii}\in\mathcal{R}_{ii}$ and $b_{ij}\in\mathcal{R}_{ij}$, $1\le i\ne j\le 2$, then*
(i) $M(a_{ii}+b_{ij})=M(a_{ii})+M(b_{ij})$;
(ii) $M^{*^{-1}}(a_{ii}+b_{ij})=M^{*^{-1}}(a_{ii})+M^{*^{-1}}(b_{ij})$.

Proof. Suppose that $M(c)=M(a_{ii})+M(b_{ij})$ for some $c\in\mathcal{R}$. For arbitrary $t_{ij}\in\mathcal{R}_{ij}$, by Lemma 7, we have

$$M^{*^{-1}}(t_{ij}c+ct_{ij})=M^{*^{-1}}(t_{ij}a_{ii}+a_{ii}t_{ij})+M^{*^{-1}}(t_{ij}b_{ij}+b_{ij}t_{ij})=M^{*^{-1}}(a_{ii}t_{ij}).$$

It follows that $t_{ij}c+ct_{ij}=a_{ii}t_{ij}$. By Lemma 4, we have $c_{ji}=0$.

Note that $t_{ij}c+ct_{ij}=t_{ij}c_{ji}+t_{ij}c_{jj}+c_{ji}t_{ij}+c_{ii}t_{ij}=t_{ij}c_{jj}+c_{ii}t_{ij}$. Therefore we have

$$t_{ij}c_{jj}+c_{ii}t_{ij}=a_{ii}t_{ij}. \tag{2}$$

Now for any $t_{jj}\in\mathcal{R}_{jj}$, using Lemma 7, we have

$$M^{*^{-1}}(t_{jj}c+ct_{jj})=M^{*^{-1}}(t_{jj}a_{ii}+a_{ii}t_{jj})+M^{*^{-1}}(t_{jj}b_{ij}+b_{ij}t_{jj})=M^{*^{-1}}(b_{ij}t_{jj}),$$

which yields that $t_{jj}c+ct_{jj}=b_{ij}t_{jj}$. It follows from Lemma 4 that $c_{jj}=0$. Moreover, equation (2) turns to be $c_{ii}t_{ij}=a_{ii}t_{ij}$, and so $c_{ii}=a_{ii}$.

Notice that $b_{ij}t_{jj}=t_{jj}c+ct_{jj}=t_{jj}c_{ji}+t_{jj}c_{jj}+c_{ij}t_{jj}+c_{jj}t_{jj}=c_{ij}t_{jj}$. Using Lemma 4 we see that $c_{ij}=b_{ij}$ $(i\neq j)$. Therefore $c=c_{ii}+c_{ij}+c_{ji}+c_{jj}=a_{ii}+b_{ij}$. Hence $M(a_{ii}+b_{ij})=M(a_{ii})+M(b_{ij})$.

By Lemma 6 we can infer that (ii) holds. □

Similarly, we can get the following result.

Lemma 9. *Let $a_{ii}\in\mathcal{R}_{ii}$ and $b_{ji}\in\mathcal{R}_{ji}$, $1\leq i\neq j\leq 2$, then*
(i) $M(a_{ii}+b_{ji})=M(a_{ii})+M(b_{ji})$;
(ii) $M^{*^{-1}}(a_{ii}+b_{ji})=M^{*^{-1}}(a_{ii})+M^{*^{-1}}(b_{ji})$.

Lemma 10. *(i) $M(a_{12}+b_{12}c_{22})=M(a_{12})+M(b_{12}c_{22})$;*
(ii) $M^{*^{-1}}(a_{12}+b_{12}c_{22})=M^{*^{-1}}(a_{12})+M^{*^{-1}}(b_{12}c_{22})$;
(iii) $M(a_{21}+b_{22}c_{21})=M(a_{21})+M(b_{22}c_{21})$;
(iv) $M^{*^{-1}}(a_{21}+b_{22}c_{21})=M^{*^{-1}}(a_{21})+M^{*^{-1}}(b_{22}c_{21})$.

Proof. Note that $a_{12}+b_{12}c_{22}=(e_1+b_{12})(a_{12}+c_{22})+(a_{12}+c_{22})(e_1+b_{12})$. We now compute

$$
\begin{aligned}
& M(a_{12}+b_{12}c_{22}) \\
= & M((e_1+b_{12})(a_{12}+c_{22})+(a_{12}+c_{22})(e_1+b_{12})) \\
= & M((e_1+b_{12})M^*M^{*^{-1}}(a_{12}+c_{22})+M^*M^{*^{-1}}(a_{12}+c_{22})(e_1+b_{12})) \\
= & M(e_1+b_{12})M^{*^{-1}}(a_{12}+c_{22})+M^{*^{-1}}(a_{12}+c_{22})M(e_1+b_{12}) \\
= & M(e_1+b_{12})M^{*^{-1}}(a_{12})+M(e_1+b_{12})M^{*^{-1}}(c_{22}) \\
& +M^{*^{-1}}(a_{12})M(e_1+b_{12})+M^{*^{-1}}(c_{22})M(e_1+b_{12}) \\
= & M((e_1+b_{12})a_{12}+a_{12}(e_1+b_{12}))+M((e_1+b_{12})c_{22}+c_{22}(e_1+b_{12})) \\
= & M(a_{12})+M(b_{12}c_{22}).
\end{aligned}
$$

Similarly, we can get $M(a_{21}+b_{22}c_{21})=M(a_{21})+M(b_{22}c_{21})$ from the fact that $a_{21}+b_{22}c_{21}=(e_1+c_{21})(a_{21}+b_{22})+(a_{21}+b_{22})(e_1+c_{21})$.

(ii) and (iv) follow from (i) and (iii) respectively by Lemma 6. □

Lemma 11. *The following are true.*
(i) $M(a_{12}+b_{12})=M(a_{12})+M(b_{12})$;
(ii) $M^{*^{-1}}(a_{12}+b_{12})=M^{*^{-1}}(a_{12})+M^{*^{-1}}(b_{12})$.

Proof. We only show (i). Suppose that $c=c_{11}+c_{12}+c_{21}+c_{22}\in\mathcal{R}$ satisfies $M(c)=M(a_{12})+M(b_{12})$. For any $t_{22}\in\mathcal{R}_{22}$, we have

$$
\begin{aligned}
M^{*^{-1}}(t_{22}c+ct_{22}) &= M^{*^{-1}}(t_{22}a_{12}+a_{12}t_{22})+M^{*^{-1}}(t_{22}b_{12}+b_{12}t_{22}) \\
&= M^{*^{-1}}(a_{12}t_{22})+M^{*^{-1}}(b_{12}t_{22}) \\
&= M^{*^{-1}}(a_{12}t_{22}+b_{12}t_{22}).
\end{aligned}
$$

Note that we apply Lemma 7 in the first equality and Lemma 10 in the last equality.

Therefore we have $t_{22}c + ct_{22} = a_{12}t_{22} + b_{12}t_{22}$. Consequently,

$$t_{22}c_{21} + t_{22}c_{22} + c_{12}t_{22} + c_{22}t_{22} = a_{12}t_{22} + b_{12}t_{22} \tag{3}$$

It follows that $t_{22}c_{21} = 0$, and so, by Lemma 4, $c_{21} = 0$.

Equation (3) also implies that $t_{22}c_{22} + c_{22}t_{22} = 0$, which yields $c_{22} = 0$.

It follows from equation (3) that $c_{12}t_{22} = a_{12}t_{22} + t_{12}t_{22}$, and so $c_{12} = a_{12} + b_{12}$.

To complete the proof it remains to show that $c_{11} = 0$. For arbitrary $t_{12} \in \mathcal{R}_{12}$, by Lemma 7, we compute

$$M^{*^{-1}}(t_{12}c + ct_{12}) = M^{*^{-1}}(t_{12}a_{12} + a_{12}t_{12}) + M^{*^{-1}}(t_{12}b_{12} + b_{12}t_{12}) = 0.$$

Then $t_{12}c + ct_{12} = 0$, consequently, $0 = t_{12}c + ct_{12} = t_{12}c_{21} + t_{12}c_{22} + c_{11}t_{12} + c_{21}t_{12} = c_{11}t_{12} = 0$. And so $c_{11} = 0$. Therefore, $c = c_{12} = a_{12} + b_{12}$. □

Lemma 12. *The following hold.*
(i) $M(a_{21} + b_{21}) = M(a_{21}) + M(b_{21})$*;*
(ii) $M^{*^{-1}}(a_{21} + b_{21}) = M^{*^{-1}}(a_{21}) + M^{*^{-1}}(b_{21})$*.*

Proof. Suppose that $M(a_{21}) + M(b_{21}) = M(c)$ for some $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{R}$.

For any $t_{22} \in \mathcal{R}_{22}$, using Lemma 7 and Lemma 10, we have

$$\begin{aligned} M^{*^{-1}}(t_{22}c + ct_{22}) &= M^{*^{-1}}(t_{22}a_{21} + a_{21}t_{22}) + M^{*^{-1}}(t_{22}b_{21} + b_{21}t_{22}) \\ &= M^{*^{-1}}(t_{22}a_{21}) + M^{*^{-1}}(t_{22}b_{21}) = M^{*^{-1}}(t_{22}a_{21} + t_{22}b_{21}) \\ &= M^{*^{-1}}(t_{22}(a_{21} + b_{21})) \end{aligned}$$

which implies that $t_{22}c + ct_{22} = t_{22}(a_{21} + b_{21})$, and so

$$t_{22}c_{21} + t_{22}c_{22} + c_{12}t_{22} + c_{22}t_{22} = t_{22}(a_{21} + b_{21}). \tag{4}$$

It follows that $t_{22}c_{22} + c_{22}t_{22} = 0$ and $c_{12}t_{22} = 0$. By Lemma 4, we have $c_{22} = 0$ and $c_{12} = 0$.

Equation (4) also implies that $t_{22}c_{21} = t_{22}(a_{21} + b_{21})$, and so $c_{21} = a_{21} + b_{21}$.

We now prove that $c_{11} = 0$. To this aim, for every $t_{21} \in \mathcal{R}_{21}$, we consider

$$M^{*^{-1}}(t_{21}c + ct_{21}) = M^{*^{-1}}(t_{21}a_{21} + a_{21}t_{21}) + M^{*^{-1}}(t_{21}b_{21} + b_{21}t_{21}) = 0.$$

Thus $t_{21}c + ct_{21} = 0$. Then we have $0 = t_{21}c + ct_{21} = t_{21}c_{11} + t_{21}c_{12} + c_{12}t_{21} + c_{22}t_{21} = t_{21}c_{11}$, and so $c_{11} = 0$. Therefore, $c = c_{21} = a_{21} + b_{21}$. The proof is complete. □

Lemma 13. *For arbitrary* $a_{11}, b_{11} \in \mathcal{R}_{11}$*, we have*
(i) $M(a_{11} + b_{11}) = M(a_{11}) + M(b_{11})$*;*
(ii) $M^{*^{-1}}(a_{11} + b_{11}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{11})$*.*

Proof. We only prove (i). Let $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{R}$ be chosen such that $M(c) = M(a_{11}) + M(b_{11})$.

For any $t_{22} \in \mathcal{R}_{22}$, by Lemma 7, we have

$$M^{*^{-1}}(t_{22}c + ct_{22}) = M^{*^{-1}}(t_{22}a_{11} + a_{11}t_{22}) + M^{*^{-1}}(t_{22}b_{11} + b_{11}t_{22}) = 0.$$

This implies that $t_{22}c + ct_{22} = 0$. By Lemma 4, we get $c_{12} = c_{21} = c_{22} = 0$.

We now show that $c_{11} = a_{11} + b_{11}$. For arbitrary $t_{12} \in \mathcal{R}_{12}$. We compute

$$\begin{aligned} & M^{*^{-1}}(t_{12}c + ct_{12}) \\ = \; & M^{*^{-1}}(t_{12}a_{11} + a_{11}t_{12}) + M^{*^{-1}}(t_{12}b_{11} + b_{11}t_{12}) \\ = \; & M^{*^{-1}}(a_{11}t_{12}) + M^{*^{-1}}(b_{11}t_{12}) = M^{*^{-1}}((a_{11} + b_{11})t_{12}). \end{aligned}$$

It follows that $t_{12}c + ct_{12} = (a_{11} + b_{11})t_{12}$. Furthermore, $c_{11}t_{12} = (a_{11} + b_{11})t_{12}$, thus $c_{11} = a_{11} + b_{11}$.

□

Similarly, we have

Lemma 14. *For arbitrary $a_{22}, b_{22} \in \mathcal{R}_{22}$, we have*
(i) $M(a_{22} + b_{22}) = M(a_{22}) + M(b_{22})$;
(ii) $M^{*^{-1}}(a_{22} + b_{22}) = M^{*^{-1}}(a_{22}) + M^{*^{-1}}(b_{22})$.

Lemma 15. *For arbitrary $a_{11} \in \mathcal{R}_{11}$ and $b_{22} \in \mathcal{R}_{22}$, the following hold.*
(i) $M(a_{11} + b_{22}) = M(a_{11}) + M(b_{22})$;
(ii) $M^{*^{-1}}(a_{11} + b_{22}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{22})$.

Proof. We only prove (i). Let $c = c_{11} + c_{12} + c_{21} + c_{22}$ be an element of $\mathcal{R}$ satisfying $M(c) = M(a_{11}) + M(a_{22})$.

For any $t_{22} \in \mathcal{R}_{22}$, we consider

$$\begin{aligned} M^{*^{-1}}(t_{22}c + ct_{22}) &= M^{*^{-1}}(t_{22}a_{11} + a_{11}t_{22}) + M^{*^{-1}}(t_{22}b_{22} + b_{22}t_{22}) \\ &= M^{*^{-1}}(t_{22}b_{22} + b_{22}t_{22}). \end{aligned}$$

This implies that $t_{22}c + ct_{22} = t_{22}b_{22} + b_{22}t_{22}$. Then we get $t_{22}c_{21} = 0$, $c_{12}t_{22} = 0$, and $t_{22}c_{cc} + c_{22}t_{22} = t_{22}b_{22} + b_{22}t_{22}$. Again, by Lemma 4, we have $c_{21} = c_{12} = 0$, and $c_{22} = b_{22}$.

To complete the proof, we need to show that $c_{11} = a_{11}$. For any $t_{12} \in \mathcal{R}_{12}$, we obtain

$$\begin{aligned} M^{*^{-1}}(t_{12}c + ct_{12}) &= M^{*^{-1}}(t_{12}a_{11} + a_{11}t_{12}) + M^{*^{-1}}(t_{12}b_{22} + b_{22}t_{12}) \\ &= M^{*^{-1}}(a_{11}t_{12}) + M^{*^{-1}}(t_{12}b_{22}) \\ &= M^{*^{-1}}(a_{11}t_{12} + t_{12}b_{22}). \end{aligned}$$

Note that in the last equality we apply Lemma 11. It follows that $t_{12}c + ct_{12} = a_{11}t_{12} + t_{12}b_{22}$, which leads to $t_{12}c_{21} + t_{12}c_{22} + c_{11}t_{12} + c_{21}t_{12} = a_{11}t_{12} + t_{12}b_{22}$, and so $c_{11}t_{12} = a_{11}t_{12}$. Therefore, $c_{11} = a_{11}$. The proof is done.

□

Lemma 16. *For arbitrary $a_{12} \in \mathcal{R}_{12}$ and $b_{21} \in \mathcal{R}_{21}$, we have*
(i) $M(a_{12} + b_{21}) = M(a_{12}) + M(b_{21})$;
(ii) $M^{*^{-1}}(a_{12} + b_{21}) = M^{*^{-1}}(a_{12}) + M^{*^{-1}}(b_{21})$.

Proof. Let $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{R}$ be chosen such that $M(c) = M(a_{12}) + M(a_{21})$.

Now for arbitrary $t_{12} \in \mathcal{R}_{12}$, we have

$$M^{*^{-1}}(t_{12}c + ct_{12}) = M^{*^{-1}}(t_{12}a_{12} + a_{12}t_{12}) + M^{*^{-1}}(t_{12}b_{21} + b_{21}t_{12}) = M^{*^{-1}}(t_{12}b_{21} + b_{21}t_{12}).$$

Therefore

$$t_{12}c + ct_{12} = t_{12}b_{21} + b_{21}t_{12},$$

i.e., $t_{12}c_{11} + t_{12}c_{22} + c_{11}t_{12} + c_{21}t_{12} = t_{12}b_{21} + b_{21}t_{12}$. This implies $t_{12}c_{22} = 0$, $c_{11}t_{12} = 0$, and $c_{21}t_{12} = b_{21}t_{12}$. Applying Lemma 4, we get $c_{22} = c_{11} = 0$ and $c_{21} = b_{21}$.

We now show that $c_{12} = a_{12}$. For any $t_{21} \in \mathcal{R}_{21}$, we obtain

$$\begin{aligned} & M^{*^{-1}}(t_{21}c + ct_{21}) \\ = \; & M^{*^{-1}}(t_{21}a_{12} + a_{12}t_{21}) + M^{*^{-1}}(t_{21}b_{21} + b_{21}t_{21}) \\ = \; & M^{*^{-1}}(t_{21}a_{12} + a_{12}t_{21}). \end{aligned}$$

This leads to

$$t_{21}c + ct_{21} = t_{21}a_{12} + a_{12}t_{21}.$$

Multiplying the above equality from the left by e_1, we arrive at $c_{12}t_{21} = a_{12}t_{21}$. And so $c_{12} = a_{12}$, as desired.

□

Lemma 17. *For any $a_{11} \in \mathcal{R}_{11}$, $b_{12} \in \mathcal{R}_{12}$, and $c_{21} \in \mathcal{R}_{21}$, we have*
(i) $M(a_{11} + b_{12} + c_{21}) = M(a_{11}) + M(b_{12}) + M(c_{21})$;
(ii) $M^{*^{-1}}(a_{11} + b_{12} + c_{21}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{12}) + M^{*^{-1}}(c_{21})$.

Proof. We pick $d = d_{11} + d_{12} + d_{21} + d_{22} \in \mathcal{R}$ such that $M(d) = M(a_{11}) + M(b_{12}) + M(c_{21})$. By Lemma 8 and Lemma 9, we have

$$M(d) = M(a_{11} + b_{12}) + M(c_{21}) \tag{5}$$

and

$$M(d) = M(a_{11} + c_{21}) + M(b_{12}). \tag{6}$$

For any $t_{21} \in \mathcal{R}_{21}$, by Lemma 7 and equation (5), we have

$$\begin{aligned} & M^{*^{-1}}(t_{21}d + dt_{21}) \\ = \; & M(^{*^{-1}}t_{21}(a_{11} + b_{12}) + (a_{11} + b_{12})t_{21}) + M^{*^{-1}}(t_{21}c_{21} + c_{21}t_{21}) \\ = \; & M^{*^{-1}}(t_{21}a_{11} + t_{21}b_{12} + b_{12}t_{21}), \end{aligned}$$

which yields that

$$t_{21}d + dt_{21} = t_{21}a_{11} + t_{21}b_{12} + b_{12}t_{21}. \tag{7}$$

Multiplying equality (7) by e_2 from the right and the left respectively, we get $t_{21}d_{12} = t_{21}b_{12}$ and $t_{21}d_{11} = t_{21}a_{11}$, and so $d_{12} = b_{12}$ and $d_{11} = a_{11}$.

We now show that $d_{21} = c_{21}$ and $d_{22} = 0$. For arbitrary $t_{12} \in \mathcal{R}_{12}$, using Lemma 7 and equality (6), we have

$$\begin{aligned} & M^{*^{-1}}(t_{12}d + dt_{12}) \\ = \ & M^{*^{-1}}(t_{12}(a_{11} + c_{21}) + (a_{11} + c_{21})t_{12}) + M^{*^{-1}}(t_{12}b_{12} + b_{12}t_{12}) \\ = \ & M^{*^{-1}}(t_{12}c_{21} + a_{11}t_{12} + c_{21}t_{12}). \end{aligned}$$

This leads to

$$t_{12}d + dt_{12} = t_{12}c_{21} + a_{11}t_{12} + c_{21}t_{12}. \tag{8}$$

Multiply equation (8) by e_1 from the right, we get $t_{12}d_{21} = t_{12}c_{21}$ and so $d_{21} = c_{21}$.

Now equality (7) turns to be $d_{22}t_{21} = 0$ and so $d_{22} = 0$. Therefore $d = a_{11}+b_{12}+c_{21}$.

By Lemma 6, we see that equality (ii) also holds. □

Similarly, we have the following

Lemma 18. *For any $a_{12} \in \mathcal{R}_{12}$, $b_{21} \in \mathcal{R}_{21}$, and $c_{22} \in \mathcal{R}_{22}$, we have*

(i) $M(a_{12} + b_{21} + c_{22}) = M(a_{12}) + M(b_{21}) + M(c_{22})$;

(ii) $M^{*^{-1}}(a_{12} + b_{21} + c_{22}) = M^{*^{-1}}(a_{12}) + M^{*^{-1}}(b_{21}) + M^{*^{-1}}(c_{22})$.

Lemma 19. *For any $a_{11} \in \mathcal{R}_{11}$, $b_{12} \in \mathcal{R}_{12}$, $c_{21} \in \mathcal{R}_{21}$, and $d_{22} \in \mathcal{R}_{22}$, the following hold.*

(i) $M(a_{11} + b_{12} + c_{21} + d_{22}) = M(a_{11}) + M(b_{12}) + M(c_{21}) + M(d_{22})$;

(ii) $M^{*^{-1}}(a_{11} + b_{12} + c_{21} + d_{22}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{12}) + M^{*^{-1}}(c_{21}) + M^{*^{-1}}(d_{22})$.

Proof. We choose $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathcal{R}$ such that

$$M(f) = M(a_{11}) + M(b_{12}) + M(c_{21}) + M(d_{22}) = M(a_{11} + d_{22}) + M(b_{12} + c_{21}).$$

We compute

$$\begin{aligned} & M^{*^{-1}}(e_1 f + f e_1) \\ = \ & M^{*^{-1}}(e_1(a_{11} + d_{22}) + (a_{11} + d_{22})e_1) + M^{*^{-1}}(e_1(b_{12} + c_{21}) + (b_{12} + c_{21})e_1) \\ = \ & M^{*^{-1}}(a_{11} + a_{11}) + M^{*^{-1}}(b_{12} + c_{21}) = M^{*^{-1}}(2a_{11}) + M^{*^{-1}}(b_{12}) + M^{*^{-1}}(c_{21}) \\ = \ & M^{*^{-1}}(2a_{11} + b_{12} + c_{21}). \end{aligned}$$

Note that in the last equality we apply Lemma 17. Then we get $e_1 f + f e_1 = 2a_{11} + b_{12} + c_{21}$. Furthermore, we have

$$2f_{11} + f_{12} + f_{21} = 2a_{11} + b_{12} + c_{21}.$$

Multiplying the above equality by e_2 from the left and the right respectively,we can infer that $f_{12} = b_{12}$, $f_{21} = c_{21}$, and $f_{11} = a_{11}$.

We need to show $f_{22} = d_{22}$ in order to complete the proof. For any $t_{12} \in \mathcal{R}_{12}$, we consider

$$
\begin{aligned}
& M^{*^{-1}}(t_{12}f + ft_{12}) \\
= \; & M^{*^{-1}}(t_{12}(a_{11} + d_{22}) + (a_{11} + d_{22})t_{12}) + M^{*^{-1}}(t_{12}(b_{12} + c_{21}) + (b_{12} + c_{21})t_{12}) \\
= \; & M^{*^{-1}}(t_{12}d_{22} + a_{11}t_{12}) + M^{*^{-1}}(t_{12}c_{21} + c_{21}t_{12}) \\
= \; & M^{*^{-1}}(t_{12}d_{22} + a_{11}t_{12} + t_{12}c_{21} + c_{21}t_{12}).
\end{aligned}
$$

Consequently,

$$t_{12}f + ft_{12} = t_{12}d_{22} + a_{11}t_{12} + t_{12}c_{21} + c_{21}t_{12},$$

this implies that $t_{12}f_{22} = t_{12}d_{22}$. Thus $f_{22} = d_{22}$. $\square$

Proof of Theorem 2 We first show that M is additive. Let $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$ be two arbitrary elements of $\mathcal{R}$. Then

$$
\begin{aligned}
& M(a + b) \\
= \; & M((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22})) \\
= \; & M(a_{11} + b_{11}) + M(a_{12} + b_{12}) + M(a_{21} + b_{21}) + M(a_{22} + b_{22}) \\
= \; & M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) + M(a_{21}) + M(b_{21}) + M(a_{22}) + M(b_{22}) \\
= \; & M(a_{11} + a_{12} + a_{21} + a_{22}) + M(b_{11} + b_{12} + b_{21} + b_{22}) \\
= \; & M(a) + M(b).
\end{aligned}
$$

That is, M is additive.

We now prove the additivity of M^*. For any $x, y \in \mathcal{R}'$, there exist $c = c_{11} + c_{12} + c_{21} + c_{22}$ and $d = d_{11} + d_{12} + d_{21} + d_{22}$ in $\mathcal{R}$ such that $c = M^*(x) + M^*(y)$ and $d = M^*(x + y)$.

For arbitrary $t_{ij} \in \mathcal{R}_{ij}$ $(1 \leq i, j \leq 2)$, using the additivity of M, we compute

$$
\begin{aligned}
& M(t_{ij}c + ct_{ij}) \\
= \; & M(t_{ij}(M^*(x) + M^*(y)) + (M^*(x) + M^*(y))t_{ij}) \\
= \; & M(t_{ij}M^*(x)) + M(t_{ij}M^*(y)) + M(M^*(x)t_{ij}) + M(M^*(y)t_{ij}) \\
= \; & M(t_{ij}M^*(x) + M^*(x)t_{ij}) + M(t_{ij}M^*(y) + M^*(y)t_{ij}) \\
= \; & M(t_{ij})x + xM(t_{ij}) + M(t_{ij})y + yM(t_{ij}) \\
= \; & M(t_{ij})(x + y) + (x + y)M(t_{ij}) \\
= \; & M(t_{ij}M^*(x + y) + M^*(x + y)t_{ij}) \\
= \; & M(t_{ij}d + dt_{ij}).
\end{aligned}
$$

Therefore,

$$t_{ij}c + ct_{ij} = t_{ij}d + dt_{ij}. \tag{9}$$

Letting $i = j = 1$ in equality (9), we get

$$t_{11}c_{11} + t_{11}c_{12} + c_{11}t_{11} + c_{21}t_{11} = t_{11}d_{11} + t_{11}d_{12} + d_{11}t_{11} + d_{21}t_{11}. \tag{10}$$

Multiply the above equation by e_1 from both sides, we get $t_{11}c_{11} + c_{11}t_{11} = t_{11}d_{11} + d_{11}t_{11}$, which implies that $c_{11} = d_{11}$.

Now equality (10) becomes

$$t_{11}c_{12} + c_{21}t_{11} = t_{11}d_{12} + d_{21}t_{11}.$$

This yields that $t_{11}c_{12} = t_{11}d_{12}$ and $c_{21}t_{11} = d_{21}t_{11}$. Then, by Lemma 4, it follows that $c_{12} = d_{12}$ and $c_{21} = d_{21}$.

To show $c_{22} = d_{22}$, we let $i = j = 2$ in equality (9) and get

$$t_{22}c_{21} + t_{22}c_{22} + c_{12}t_{22} + c_{22}t_{22} = t_{22}d_{21} + t_{22}d_{22} + d_{12}t_{22} + d_{22}t_{22},$$

which leads to $t_{22}c_{22} + c_{22}t_{22} = t_{22}d_{22} + d_{22}t_{22}$, and so $c_{22} = d_{22}$. Consequently, $c = d$, hence $M^*(x + y) = M^*(x) + M^*(y)$, which completes the proof.

For the case of Jordan elementary maps on prime rings we have the following result.

Corollary 20. *Let $\mathcal{R}$ be a 2-torsion free prime ring containing a nontrivial idempotent e_1, and $\mathcal{R}'$ be an arbitrary ring. Suppose that $e_2ae_2be_2 + e_2be_2ae_2 = 0$ for each $b \in \mathcal{R}$ implies $e_2ae_2 = 0$. Let $M\colon \mathcal{R} \to \mathcal{R}'$ and $M^*\colon \mathcal{R}' \to \mathcal{R}$ be two surjective maps such that*

$$\begin{cases} M(aM^*(x) + M^*(x)a) = M(a)x + xM(a), \\ M^*(M(a)x + xM(a)) = aM^*(x) + M^*(x)a \end{cases}$$

for all $a \in \mathcal{R}$, $x \in \mathcal{R}'$. Then both M and M^* are additive.

Proof. Since $\mathcal{R}$ is prime, it is easy to check that condition (i) of Theorem 2 holds true. Now the proof goes directly. □

In particular, if a prime ring has an identity element, then we have

Corollary 21. *Let $\mathcal{R}$ be a 2-torsion free unital prime ring containing a nontrivial idempotent e_1, and $\mathcal{R}'$ be an arbitrary ring. Suppose that $M\colon \mathcal{R} \to \mathcal{R}'$ and $M^*\colon \mathcal{R}' \to \mathcal{R}$ are two surjective maps such that*

$$\begin{cases} M(aM^*(x) + M^*(x)a) = M(a)x + xM(a), \\ M^*(M(a)x + xM(a)) = aM^*(x) + M^*(x)a \end{cases}$$

for all $a \in \mathcal{R}$, $x \in \mathcal{R}'$. Then both M and M^* are additive.

We complete this note by considering Jordan elementary maps on standard operator algebras.

Corollary 22. *Let $\mathcal{A}$ be a standard operator algebra on a Banach space of dimension greater than 1, and $\mathcal{R}$ be an arbitrary ring. Suppose that $M\colon \mathcal{A} \to \mathcal{R}$ and $M^*\colon \mathcal{R} \to \mathcal{A}$ are surjective maps such that*

$$\begin{cases} M(aM^*(x) + M^*(x)a) = M(a)x + xM(a), \\ M^*(M(a)x + xM(a)) = aM^*(x) + M^*(x)a \end{cases}$$

for all $a \in \mathcal{A}$, $x \in \mathcal{R}$. Then both M and M^* are additive.

Proof. Note that, by Lemma 2 in [4], we see that $\mathcal{A}$ satisfies conditions (i) and (ii) of Theorem 2. Now, the proof follows easily. □

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